Advanced Mathematical Models & Applications
Vol.8, No.1, 2023, pp.19-33

# ALGEBRAIC STRUCTURE IN CERTAIN GENERALIZED FUZZY SETS*

**Miguel Cruz Ramírez[1†], Juan E. Nápoles Valdés[2], Alejandro Figueredo López[1], Natavan Allahverdiyeva[3]**

[1]University of Holguín, Holguín, Cuba
[2]National University of the Northeast, Corrientes, Argentina
[3]Sumgayit State University, Sumgayit, Azerbaijan

**Abstract.** The present work is related to a certain extension of intuitionist fuzzy sets, endowed with an addition operation, a scalar product, and a preorder relation. We show that this structure is a bounded left $\mathbb{R}_{\geqslant 0}$-semimodule. We also discuss some metric properties and develop a new approach to the concept of mean, the application of which is illustrated by an example.

**Keywords**: A-IFS Atanassov's intuitionistic fuzzy set, fuzzy abelian monoids, fuzzy left $R$-semimodules, fuzzy preordered sets, fuzzy mean.

**AMS Subject Classification:** 08A72, 18B35.

[†]**Corresponding author:** Miguel Cruz Ramírez, University of Holguín, Department of Mathematics, Holguín, Cuba, Tel.: +53 59383911, e-mail: *cruzramirezmiguel@gmail.com*

*Received: 24 July 2022; Revised: 13 October 2022; Accepted: 6 December 2022; Published: 13 April 2023.*

## 1 Introduction

Fuzzy sets were introduced by Zadeh as a generalization of classical set theory (Zadeh, 1965). A fuzzy set is characterized by a membership function $\mu$, which takes value from a crisp set to a unit interval $I = [0, 1]$, in order to indicate the degree of belongingness of each element to the set under consideration. Numerous extensions and special cases of this concept have emerged in recent years, such as, bipolar-valued fuzzy sets, complex fuzzy sets, grey sets, interval-valued fuzzy sets, typical hesitant fuzzy sets, shadow sets, among others (Bustince et al., 2016).

An important extension is due to Atanassov, and consists in the addition of a non-membership function $\nu$, satisfying the condition $0 \leqslant \mu + \nu \leqslant 1$ (Atanassov, 1986). The notion of A-IFS (Atanassov's intuitionistic fuzzy set) provides a flexible framework to interpret vagueness and uncertainty, with hesitation margin $\pi = 1 - \mu - \nu$. The minimal value $\pi = 0$ indicates absence of uncertainty, which characterizes a fuzzy set in the Zadeh sense. The A-IFS concept is useful for modeling many types of practical problems (Atanassov, 2012). For example, in electoral polls, a candidate $x$ may have 45% of intentions to vote in favor, 31% against, while the margin of 24% of undecided voters remains. This fact can be expressed by means of the triad $\langle x, 0.45, 0.31\rangle$, where $0 \leqslant \mu + \nu = 0.45 + 0.31 = 0.76 \leqslant 1$, and $\pi = 0.24$. Considering $N$ respondents, if each survey opinion produces a value $\mu_i$ and $\nu_i$ then, for each candidate $x$, the expression $\langle x, \frac{1}{N}\sum_{i=1}^{N}\mu_i(x), \frac{1}{N}\sum_{i=1}^{N}\nu_i(x)\rangle$ based on usual averages makes sense and meaning. Now, since the variable $x$ has a nominal scale, it only remains to establish some kind of mechanism to compare the options of each candidate.

*****How to cite (APA)**: Ramírez, M.C., Nápoles Valdés, J.E., Figueredo López, A. & Allahverdiyeva, N. (2023). Algebraic structure in certain generalized fuzzy sets. *Advanced Mathematical Models & Applications*, *8*(1), 19-33.

On the other hand, several experts could express their estimate regarding a specific date, where presumably a certain event should occur with greater probability. In this case, the forecasting problem could be represented using non-negative real number intervals, on which it is possible to consider two functions: one of membership and the other of non-membership. This type of problem can be modeled by means of the so-called triangular fuzzy numbers, but in the A-IFS sense (Seikh et al., 2013).

There are also situations that do not satisfy the conditions of the A-IFS model. Indeed, an individual can evaluate with $\frac{\sqrt{3}}{2}$ the degree to which an alternative $a_i$ satisfies a certain criterion $C_j$ and, at the same time, in another item evaluate with $\frac{1}{2}$ the degree to which such alternative does not satisfy the mentioned criterion. The deficiency does not lie in the design of the attitude scale, nor in the degree of validity of the response, but in the underlying mathematical model. To overcome this obstacle, Yager introduced Pythagorean fuzzy sets under the $\mu^2 + \nu^2 \leqslant 1$ condition (Yager, 2013), which does fit the type of responses mentioned above.

Yager's approach serves to moderate relatively exaggerated responses on attitude scales, but there are also cases where the individual expresses remarkably conservative responses with observed values that satisfy the A-IFS model, but are too small. For this case, there is also a model that uses radicals instead of squares, namely, the intuitionistic fuzzy sets of root-type where $\frac{1}{2}\sqrt{\mu} + \frac{1}{2}\sqrt{\nu} \leqslant 1$ (Srinivasan & Palaniappan, 2012). In the present work we consider a generalization of the A-IFS, which encompasses both the Pythagorean and the root type cases (Jamkhaneh & Nadarajah, 2015; Jamkhaneh & Garg, 2018). In particular, we analyze the algebraic nature of this class of generalized fuzzy set, once it has been endowed with an addition operation, a dot product and a certain preorder relation. We show that this structure is a bounded left $\mathbb{R}_{\geqslant 0}$-semimodule. We also analyze some metric properties of $\delta$-fuzzy sets, which allows us to introduce a new approach to the concept of generalized fuzzy mean. Finally, we present an example where the calculation of a $\delta$-fuzzy mean is applied.

## 2 An algebraic approach to $\delta$-fuzzy sets

Next we will present the concept of the $\delta$-fuzzy set, developed by Jamkhaneh and other authors, as well as the main properties (Jamkhaneh & Nadarajah, 2015; Jamkhaneh & Garg, 2018). Our approach will follow a predominantly algebraic path. As Milošević et al. point out (Milošević et al., 2021), this generalization can be approached in other very similar ways, such as the case of Yager's $q$-th rung orthopair fuzzy sets (Yager, 2017), and the case of A-IFS of $n$th type (Atanassov & Vassilev, 2018).

**Definition 1.** *Let $\delta \in \mathbb{R}_{>0}$ and let $X = \{x_i\}_{i\in I}$ a nonempty universe of discourse. A set $\widetilde{A}$ is called $\delta$-fuzzy if there are two aggregation functions $\mu_{\widetilde{A}} : X \to [0,1]$ and $\nu_{\widetilde{A}} : X \to [0,1]$, with $0 \leqslant \mu^{\delta}_{\widetilde{A}}(x) + \nu^{\delta}_{\widetilde{A}}(x) \leqslant 1$ $(\forall x \in X)$, such that*

$$\widetilde{A} = \{\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\rangle \,|\, x \in X\}.$$

*The functions $\mu_{\widetilde{A}}(x)$ y $\nu_{\widetilde{A}}(x)$ denote, respectively, the degree of membership and degree of non-membership of element $x$ to the set $\widetilde{A}$. In this case, it is said that the expression $\pi_{\widetilde{A}}(x) = [1 - \mu^{\delta}_{\widetilde{A}}(x) - \nu^{\delta}_{\widetilde{A}}(x)]^{\frac{1}{\delta}}$ is called the degree of hesitation or indeterminacy of $x$ to $\widetilde{A}$.*

In a brief way, we will use the abbreviated notation $\langle \mu_{\widetilde{A}}, \nu_{\widetilde{A}}\rangle$ to refer to the set $\widetilde{A} = \{\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\rangle \,|\, x \in X\}$. The above definition gives rise to several particular concepts, already used previously in the development of fuzzy approaches. For example, if $\delta = 1$, the concept of A-IFS results directly. In addition, if $\nu_{\widetilde{A}} = 1 - \mu_{\widetilde{A}}$, then Zadeh's seminal definition of a fuzzy set is obtained, where $\pi_{\widetilde{A}} = 0$. If we also consider $\mu_{\widetilde{A}} = \chi_A : X \to \{0,1\}$ as the characteristic function of $A$ in $X$, that is,

$$\mu_{\widetilde{A}}(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

then a crisp interpretation of the concept of set results. Cases $\delta = 2$ and $\delta = 3$ coincide with the definitions of Pythagorean fuzzy set (Yager, 2013) and Fuzzy sets of Third Type (Begum & Srinivasan, 2017), respectively. The intuitionistic fuzzy sets of root-type (Srinivasan & Palaniappan, 2012) does not conform to the general notion considered for $\delta = \frac{1}{2}$ , which has already been indicated previously (Atanassov & Vassilev, 2018). It is clear that if $\mu, \nu, \delta_1, \delta_2$ are real numbers such that $\mu, \nu \in [0, 1]$ and $0 < \delta_1 < \delta_2$, then from the condition $0 \leqslant \mu^{\delta_1} + \nu^{\delta_1} \leqslant 1$ it follows that $0 \leqslant \mu^{\delta_2} + \nu^{\delta_2} \leqslant 1$. Therefore, the following property takes place.

**Theorem 1.** *If $\delta_1 < \delta_2$, then every set $\delta_1$-fuzzy is also $\delta_2$-fuzzy.*

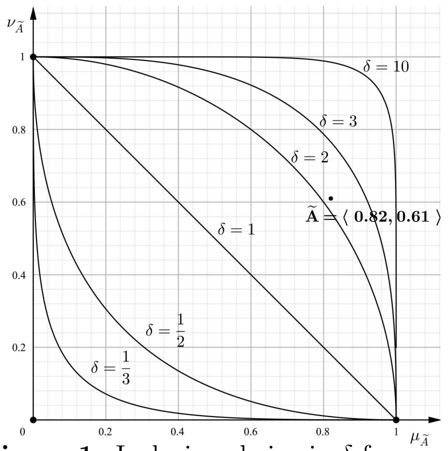


**Figure 1:** Inclusion chains in $\delta$-fuzzy sets

From this theorem, it can be seen that intuitionistic fuzzy sets of root-type constitute a proper subset of A-IFS. In turn, A-IFS are fully included in Pythagorean fuzzy sets and so on, forming an inclusion chains as illustrated in Figure 1, where the unit square $[0, 1] \times [0, 1]$ constitutes the supremum set. As an example, point $\widetilde{A} = \langle 0.82, 0.61\rangle$ represents the generalized intuitionist component for a certain $x$-value of the universe of discourse. Its coordinates are enclosed by a kind of lunula, delimited by the curves $\mu^2_{\widetilde{A}} + \nu^2_{\widetilde{A}} = 1$ and $\mu^3_{\widetilde{A}} + \nu^3_{\widetilde{A}} = 1$, within the first quadrant. The components of $\widetilde{A}$ were found experimentally in an empirical study related to subjective perception, for the case of an individual and under specific conditions (Cruz & Cables, 2021). Obviously the Pythagorean model does not comprise point $\widetilde{A}$, therefore it is necessary to determine the minimum value of $\delta$, so that the mathematical model is adequate and describes well each evidence, in all or at least in most of the empirical observations.

The operations of union and intersection of $\delta$-fuzzy sets can be defined in a generalized way, through triangular norms and triangular conorms (Deschrijver & Kerre, 2002).

**Definition 2.** *For two $\delta$-fuzzy sets $\widetilde{A}$ and $\widetilde{B}$, the generalized union and intersection operators are defined respectively as follows:*

$$\widetilde{A} \vee \widetilde{B} = \langle S(\nu_{\widetilde{A}}, \nu_{\widetilde{B}}), T(\mu_{\widetilde{A}}, \mu_{\widetilde{B}})\rangle,$$

$$\widetilde{A} \wedge \widetilde{B} = \langle T(\mu_{\widetilde{A}}, \mu_{\widetilde{B}}), S(\nu_{\widetilde{A}}, \nu_{\widetilde{B}})\rangle,$$

*where $T$ is any t-norm and $S$ is any t-conorm.*

Based on this concept, it is possible to introduce operations between $\delta$-fuzzy sets. One of them consists in developing a certain kind of sum, where each element of $X$ has membership and non-membership values dependent on $\mu$ and $\nu$ in the original sets. This is the case in the following definition.

**Definition 3.** *Let $\widetilde{A}$ and $\widetilde{B}$, two $\delta$-fuzzy sets of the universe of discourse $X$. The sum $\delta$-fuzzy of $\widetilde{A}$ and $\widetilde{B}$ is defined as follows*

$$\widetilde{A} \oplus \widetilde{B} = \langle (\mu^{\delta}_{\widetilde{A}} + \mu^{\delta}_{\widetilde{B}} - \mu^{\delta}_{\widetilde{A}}\mu^{\delta}_{\widetilde{B}})^{\frac{1}{\delta}},\ \nu_{\widetilde{A}}\nu_{\widetilde{B}} \rangle.$$

It can be verified that the function $S : [0,1] \times [0,1] \to [0,1]$, defined by $S(x,y) = (x^{\delta} + y^{\delta} - x^{\delta}y^{\delta})^{\frac{1}{\delta}}$ satisfies the axioms of a $t$-conorm:

*a)* Commutativity: $S(x,y) = S(y,x)$

*b)* Associativity: $S(x, S(y,z)) = S(S(x,y), z)$

*c)* Monotonicity: $S(x,y) \leqslant S(z,t)$ si $x \leqslant z$ y $y \leqslant t$

*d)* Boundary condition: $S(1,1) = 1$ y $S(x,0) = x$.

Also, the function $T : [0,1] \times [0,1] \to [0,1]$ defined by $T(x,y) = xy$ is a $t$-norm, since it satisfies the same previous axioms, except the fourth where 1 is exchanged for 0, i.e., $T(0,0) = 0$ and $T(x,1) = x$. Therefore, the set $\widetilde{A} \oplus \widetilde{B}$, constitutes a particular type of generalized $\delta$-fuzzy union. It can be noted that $\oplus$ is a commutative and closed operation in the powerset $\widetilde{\mathcal{P}}(X)$. In addition, a brief arithmetic work reveals that for all $\delta$-fuzzy sets $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \widetilde{\mathcal{P}}(X)$, this operation also fulfills the associative property $(\widetilde{A} \oplus \widetilde{B}) \oplus \widetilde{C} = \widetilde{A} \oplus (\widetilde{B} \oplus \widetilde{C})$. Furthermore, there are exactly three idempotent elements: $\langle 0,1\rangle, \langle 0,0\rangle, \langle 1,0\rangle \in \widetilde{\mathcal{P}}(X)$, of which the set $\widetilde{\mathcal{O}} = \langle 0,1\rangle$, satisfies that for all $\delta$-fuzzy set $\widetilde{A} \in \widetilde{\mathcal{P}}(X)$ it follows that $\widetilde{\mathcal{O}} \oplus \widetilde{A} = \widetilde{A} \oplus \widetilde{\mathcal{O}} = \widetilde{A}$. With the exception of $\widetilde{\mathcal{O}}$, all elements in $\widetilde{\mathcal{P}}(X)$ have no invertible element. In short, the following theorem holds.

**Theorem 2.** *The algebraic structure $(\widetilde{\mathcal{P}}(X), \oplus)$ constitutes an abelian monoid, that is, a commutative semigroup provided with the identity element $\widetilde{\mathcal{O}} = \langle 0,1\rangle$.*

Similarly, it is possible to define a product by reasoning in a dual way, that is $\widetilde{A} \otimes \widetilde{B} = \langle \mu_{\widetilde{A}}\mu_{\widetilde{B}}, (\nu^{\delta}_{\widetilde{A}} + \nu^{\delta}_{\widetilde{B}} - \nu^{\delta}_{\widetilde{A}}\nu^{\delta}_{\widetilde{B}})^{\frac{1}{\delta}} \rangle$, as well as inclusion relations and also union, intersection and complement operations (Jamkhaneh & Nadarajah, 2015; Jamkhaneh & Garg, 2018). The structure $(\widetilde{\mathcal{P}}(X), \otimes)$ is also an abelian monoid, whose identity element is $\widetilde{\mathcal{I}} = \langle 1,0\rangle$. In general, neither of these operations $\oplus$ and $\otimes$ distributes with respect to the other. It can even be verified that $\widetilde{A} \otimes (\widetilde{B} \oplus \widetilde{C}) = (\widetilde{A} \otimes \widetilde{B}) \oplus (\widetilde{A} \otimes \widetilde{C})$ iff $\widetilde{A} = \widetilde{\mathcal{I}}$, or if at least one of the $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ values is equal to $\widetilde{\mathcal{O}}$. In a similar way to the Pythagorean case (Peng & Yang, 2015), the complementation can be defined as $\langle \mu, \nu\rangle^{\mathrm{c}} = \langle \nu, \mu\rangle$ (Jamkhaneh & Nadarajah, 2015), from which Morgan's laws are valid in delta-fuzzy sets: $(\widetilde{A} \oplus \widetilde{B})^{\mathrm{c}} = \widetilde{A}^{\mathrm{c}} \otimes \widetilde{B}^{\mathrm{c}}$ and $(\widetilde{A} \otimes \widetilde{B})^{\mathrm{c}} = \widetilde{A}^{\mathrm{c}} \oplus \widetilde{B}^{\mathrm{c}}$. In the present work, we do not address the relationships between $\oplus$ and $\otimes$, but rather delve into the $(\widetilde{\mathcal{P}}(X), \oplus)$ monoid, provided with a dot product.

**Definition 4.** *The mapping $* : \mathbb{R}_{\geqslant 0} \times \widetilde{\mathcal{P}}(X) \to \widetilde{\mathcal{P}}(X)$, $(r, \widetilde{A}) \to r * \widetilde{A}$, where*

$$r * \widetilde{A} = \langle [1 - (1 - \mu^{\delta}_{\widetilde{A}})^{r}]^{\frac{1}{\delta}}, \nu^{r} \rangle$$

*is a dot product in a $\widetilde{\mathcal{P}}(X)$.*

This definition is consistent, since $r * \widetilde{A}$ is also a $\delta$-fuzzy set. Indeed, as $1 - \mu^{\delta}_{\widetilde{A}} \geqslant \nu^{\delta}_{\widetilde{A}}$, we have that $0 \leqslant \mu^{\delta}_{r*\widetilde{A}} + \nu^{\delta}_{r*\widetilde{A}} = 1 - (1 - \mu^{\delta}_{\widetilde{A}})^{r} + \nu^{r\delta}_{\widetilde{A}} \leqslant 1 - (\nu^{\delta}_{\widetilde{A}})^{r} + \nu^{r\delta}_{\widetilde{A}} = 1$. On the other hand, it is well known that $(\mathbb{R}_{\geqslant 0}, +, \cdot)$ is an Abelian semiring, for which the following properties are also verified for all $r, s \in \mathbb{R}_{\geqslant 0}$, and all $\widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{P}}(X)$:

*a)* $(rs) * \widetilde{A} = r * (s * \widetilde{A})$

*b)* $r * (\widetilde{A} \oplus \widetilde{B}) = (r * \widetilde{A}) \oplus (r * \widetilde{B})$

*c)* $(r+s) * \widetilde{A} = (r * \widetilde{A}) \oplus (s * \widetilde{A})$

*d)* $1 * \widetilde{A} = \widetilde{A}$

*e)* $r * \widetilde{\mathcal{O}} = \widetilde{\mathcal{O}} = 0 * \widetilde{\mathcal{O}}$

These five properties constitute the axioms defined by Golan for the structure of an left $R$-semimodule (see Golan, 2003, p. 101), for a given abelian monoid and with respect to a certain semiring $R$. Properties (a), (d) and (e) follow directly from the Definition 4. Next, we will prove property (c).

$$\begin{aligned}(r * \widetilde{A}) \oplus (s * \widetilde{A}) &= \langle [1-(1-\mu_{\widetilde{A}}^{\delta})^{r}]^{\frac{1}{\delta}}, \nu_{\widetilde{A}}^{r}\rangle \oplus \langle [1-(1-\mu_{\widetilde{A}}^{\delta})^{s}]^{\frac{1}{\delta}}, \nu_{\widetilde{A}}^{s}\rangle \\ &= \langle \big\{1-(1-\mu_{\widetilde{A}}^{\delta})^{r} + 1-(1-\mu_{\widetilde{A}}^{\delta})^{s} - [1-(1-\mu_{\widetilde{A}}^{\delta})^{r}][1-(1-\mu_{\widetilde{A}}^{\delta})^{s}]\big\}^{\frac{1}{\delta}}, \nu_{\widetilde{A}}^{r+s}\rangle \\ &= \langle [1-(1-\mu_{\widetilde{A}}^{\delta})^{r+s}]^{\frac{1}{\delta}}, \nu_{\widetilde{A}}^{r+s}\rangle \\ &= (r+s) * \widetilde{A}.\end{aligned}$$

Property (b) is verified using similar resources. Consequently, we arrive at the following theorem:

**Theorem 3.** *The abelian monoid $(\widetilde{\mathcal{P}}(X), \oplus)$ provided with the dot product $*$ constitutes an left $\mathbb{R}_{\geqslant 0}$-semimodule.*

To conclude this section, we will address the problem related to the ordering of $\delta$-fuzzy sets. We will start from the following definition.

**Definition 5.** *Let $\widetilde{A}$ and $\widetilde{B}$ be two $\delta$-fuzzy sets. We will say that $\widetilde{A} \prec \widetilde{B}$ iff there are a $\delta$-fuzzy set $\widetilde{M}$ so that $\widetilde{A} \oplus \widetilde{M} = \widetilde{B}$.*

This kind of relation satisfies the properties of reflexivity and transitivity in any monoid and is called canonical preordering set (Gondran & Minoux, 1984) and also difference ordered set (Golan, 2003). In the $\widetilde{\mathcal{P}}(X)$ set, the relation does not induce a partial order, since the antisymmetry fails in a general way. Furthermore, in Definition 5, if the set $\{\widetilde{M} \mid \widetilde{A} \oplus \widetilde{M} = \widetilde{B}\}$ is either empty or a singleton, whenever $\widetilde{A} \neq \widetilde{B}$, then the relation $\prec$ constitutes a weak uniquely difference ordered (WUDO in Golan, 2003). This notion allows, in Golan's sense, to define the element $\widetilde{B} \ominus \widetilde{A}$ by setting

$$\widetilde{B} \ominus \widetilde{A} = \begin{cases} \left\langle \left(\frac{\mu_{\widetilde{B}}^{\delta} - \mu_{\widetilde{A}}^{\delta}}{1-\mu_{\widetilde{A}}^{\delta}}\right)^{\frac{1}{\delta}},\ \frac{\nu_{\widetilde{B}}}{\nu_{\widetilde{A}}} \right\rangle & \text{if} \qquad \mu_{\widetilde{B}} \geqslant \mu_{\widetilde{A}},\ \nu_{\widetilde{A}} \neq 0,\ \nu_{\widetilde{B}} \leqslant \min\limits_{x\in X}\left\{\nu_{\widetilde{A}}, \dfrac{\nu_{\widetilde{A}}\pi_{\widetilde{B}}}{\pi_{\widetilde{A}}}\right\}, \\ \widetilde{\mathcal{O}} = \langle 0, 1\rangle & \text{if not.} \end{cases}$$

This difference set has already been previously analyzed in the Pythagorean case (Peng & Yang, 2015), where this set is unique and satisfies $\widetilde{B} \ominus \widetilde{\mathcal{O}} = \widetilde{B}$ and the cancellation property $(\widetilde{B} \ominus \widetilde{A}) \oplus \widetilde{A} = \widetilde{B}$ whenever $\widetilde{A} \prec \widetilde{B}$. In general, the following statement is valid.

**Theorem 4.** *The binary relation $\prec$ is a preorder, compatible with $(\widetilde{\mathcal{P}}(X), \oplus, *)$ as right $\mathbb{R}_{\geqslant 0}$-semimodule.*

*Proof.* The algebraic structure $(\widetilde{\mathcal{P}}(X), \oplus)$ is a monoid, therefore the associativity and the existence of the identity element are verified. Since $\widetilde{A} \oplus \mathcal{O} = \widetilde{A}$ we have $\widetilde{A} \prec \widetilde{A}$, that is, $\prec$ is a reflexive relation. If $\widetilde{A} \prec \widetilde{B}$ and $\widetilde{B} \prec \widetilde{C}$, it follows respectively that there exist $\widetilde{M}$ and $\widetilde{N}$ such that $\widetilde{A} \oplus \widetilde{M} = \widetilde{B}$ and $\widetilde{B} \oplus \widetilde{N} = \widetilde{C}$. Then, it turns out that $\widetilde{C} = \widetilde{B} \oplus \widetilde{N} = (\widetilde{A} \oplus \widetilde{M}) \oplus \widetilde{N} = \widetilde{A} \oplus (\widetilde{M} \oplus \widetilde{N})$ and it is enough to choose $\widetilde{S} = \widetilde{M} \oplus \widetilde{N}$ to conclude that $\widetilde{A} \prec \widetilde{C}$, so the relation is also transitive. In short, the binary relation $\prec$ is a preorder. On the other hand, to test compatibility with the closed addition and dot product, the following properties must be verified (Golan, 2003, p. 137):

*a)* $\widetilde{A} \prec \widetilde{B} \Rightarrow \widetilde{A} \oplus \widetilde{C} \prec \widetilde{B} \oplus \widetilde{C}, \qquad \forall \widetilde{A}, \widetilde{B}, \widetilde{C} \in \widetilde{\mathcal{P}}(X)$

*b)* $(\widetilde{A} \prec \widetilde{B} \wedge 0 \leqslant r) \Rightarrow r * \widetilde{A} \prec r * \widetilde{B}, \qquad \forall \widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{P}}(X), \forall r \in \mathbb{R}_{\geqslant 0}$

*c)* $(r \leqslant s \wedge \widetilde{\mathcal{O}} \prec \widetilde{A}) \Rightarrow r * \widetilde{A} \prec s * \widetilde{A}, \qquad \forall r, s \in \mathbb{R}_{\geqslant 0}, \forall \widetilde{A} \in \widetilde{\mathcal{P}}(X).$

The first case results directly from the definition of $\prec$ and the commutativity of $\oplus$. Indeed, if $\widetilde{A} \prec \widetilde{B}$ then, there exist $\widetilde{M}$ such that $\widetilde{A} \oplus \widetilde{M} = \widetilde{B}$ and thereupon $\widetilde{B} \oplus \widetilde{C} = (\widetilde{A} \oplus \widetilde{M}) \oplus \widetilde{C} = (\widetilde{A} \oplus \widetilde{C}) \oplus \widetilde{M}$. Therefore, $\widetilde{A} \oplus \widetilde{C} \prec \widetilde{B} \oplus \widetilde{C}$. The second case follows from the distributivity of the scalar product with respect to the sum of $\delta$-fuzzy sets. From the relation $\widetilde{A} \prec \widetilde{B}$, the existence of $\widetilde{M}$ is deduced such that $\widetilde{A} \oplus \widetilde{M} = \widetilde{B}$. So it turns out that $r * \widetilde{B} = r * (\widetilde{A} \oplus \widetilde{M}) = (r * \widetilde{A}) \oplus (r * \widetilde{M})$, since $r \geqslant 0$. Then, we have that $r * \widetilde{A} \prec r * \widetilde{B}$, considering the $\delta$-fuzzy set $\widetilde{S} = r * \widetilde{M}$. In the third case, we have that there exists $t \geqslant 0$ such that $s = r + t$. Then $s * \widetilde{A} = (r+t) * \widetilde{A} = (r * \widetilde{A}) \oplus (t * \widetilde{A})$, which implies that $r * \widetilde{A} \prec s * \widetilde{A}$, since $\widetilde{S} = t * \widetilde{A}$ is a $\delta$-fuzzy set. $\square$

Obviously, from the equality $\widetilde{\mathcal{O}} \oplus \widetilde{A} = \widetilde{A}$ we deduce that $\widetilde{\mathcal{O}} \prec \widetilde{A}$. After considering $f(\widetilde{X}) = \widetilde{\mathcal{I}} \ominus \widetilde{X}$, we obtain a sort of $\delta$-fuzzy characteristic function

$$f(\widetilde{X}) = \begin{cases} \widetilde{\mathcal{I}} & \text{if} \quad \widetilde{X} \neq \widetilde{\mathcal{I}} \\ \widetilde{\mathcal{O}} & \text{if} \quad \widetilde{X} = \widetilde{\mathcal{I}}. \end{cases}$$

Note that $\widetilde{A} \oplus (\widetilde{\mathcal{I}} \ominus \widetilde{A}) = \widetilde{A} \oplus f(\widetilde{A}) = \widetilde{A}1 \oplus \widetilde{\mathcal{I}} = \widetilde{\mathcal{I}}$ so, taking into account the existence of the $\delta$-fuzzy set $\widetilde{S} = f(\widetilde{A})$, we see that $\widetilde{A} \prec \widetilde{\mathcal{I}}$. Therefore, we have a bounded preorder $\widetilde{\mathcal{O}} \prec \widetilde{A} \prec \widetilde{\mathcal{I}}$, for all $\delta$-fuzzy set $\widetilde{A} \in \widetilde{\mathcal{P}}(X)$. This fact is analogous to a property of the set of natural numbers, where the partial order induced by the divisibility relation is bounded, precisely by the identity elements for the product and the sum respectively, that is: $1 \prec n \prec 0$, for all $n \in \mathbb{N}$.

## 3 Some metric properties of $\delta$-fuzzy sets

In this section we present a concept of distance for $\delta$-fuzzy sets, which generalize some particular cases described in the literature. First, we will restrict our analysis to a real universe of discourse, where we differentiate between discrete and continuous cases.

**Definition 6.** *The distance between two discrete $\delta$-fuzzy sets $\widetilde{A}$ and $\widetilde{B}$, of the universe of discourse $X = \{x_i\}_{1 \leqslant i \leqslant m} \subset \mathbb{R}$, is defined by the expression*

$$d(\widetilde{A}, \widetilde{B}) = \frac{1}{2m} \sum_{i=1}^{m} (|\mu^{\delta}_{\widetilde{A}}(x_i) - \mu^{\delta}_{\widetilde{B}}(x_i)| + |\nu^{\delta}_{\widetilde{A}}(x_i) - \nu^{\delta}_{\widetilde{B}}(x_i)| + |\pi^{\delta}_{\widetilde{A}}(x_i) - \pi^{\delta}_{\widetilde{B}}(x_i)|).$$

The expression $d(\widetilde{A}, \widetilde{B})$ is a non-negative and also symmetric with respect to its variables. We have $d(\widetilde{A}, \widetilde{B}) \neq 0$ iff there exists a $i$ such that $\mu_{\widetilde{A}}(x_i) \neq \mu_{\widetilde{B}}(x_i)$, or $\nu_{\widetilde{A}}(x_i) \neq \nu_{\widetilde{B}}(x_i)$, or $\pi_{\widetilde{A}}(x_i) \neq \pi_{\widetilde{B}}(x_i)$, then $d(\widetilde{A}, \widetilde{B}) \neq 0$ and that is, $d(\widetilde{A}, \widetilde{B}) = 0 \Leftrightarrow \widetilde{A} = \widetilde{B}$. Triangular inequality is a direct consequence of using the property $|x - y| + |y - z| \geqslant |x - z|$, when comparing each modular expression for the membership, non-membership, and indeterminacy functions, respectively. In summary, $d$ satisfies the distance axioms, so the Definition 6 is consistent. Note that for $\delta = 1$, results in a Hamming distance for A-IFS (Szmidt, 2014). If $\delta = 2$, the result is a normalized distance between Pythagorean fuzzy sets (Peng & Yang, 2015; Zhang & Xu, 2014; and cf. Ejegwa, 2018).

On the other hand, for the case of two Riemann integrable functions $\mu(x)$ and $\nu(x)$ on a continuum $X$ as discursive universe, we have the following analogous definition that generalizes the one corresponding to A-IFS (cf. Tcvetkov et al., 2009).

**Definition 7.** *The distance between two $\delta$-fuzzy sets $\widetilde{A}$ and $\widetilde{B}$, of the universe of discourse $X=[a,b]\subset\mathbb{R}$, is defined by the expression*

$$d(\widetilde{A},\widetilde{B})=\frac{1}{2(b-a)}\int_a^b(|\mu^\delta_{\widetilde{A}}(x)-\mu^\delta_{\widetilde{B}}(x)|+|\nu^\delta_{\widetilde{A}}(x)-\nu^\delta_{\widetilde{B}}(x)|+|\pi^\delta_{\widetilde{A}}(x)-\pi^\delta_{\widetilde{B}}(x)|)dx$$

Here are some properties that are derived from the concept of distance. We will only refer to the case in which $X$ is a real interval, since the discrete case develops in an analogous way. Theorem 5 reveals that the distance between any two triangular $\delta$-fuzzy numbers is always bounded.

**Theorem 5.** *Let $\widetilde{A}$ and $\widetilde{B}$ be two $\delta$-fuzzy sets, defined on the interval $X=[a,b]\subset\mathbb{R}$. Then $d(\widetilde{A},\widetilde{B})\leqslant 1$.*

*Proof.* Successively it turns out that

$$\begin{aligned}
d(\widetilde{A},\widetilde{B}) &= \frac{1}{2(b-a)}\int_a^b\left\{|\mu^\delta_{\widetilde{A}}(x)-\mu^\delta_{\widetilde{B}}(x)|+|\nu^\delta_{\widetilde{A}}(x)-\nu^\delta_{\widetilde{B}}(x)|+|\pi^\delta_{\widetilde{A}}(x)-\pi^\delta_{\widetilde{B}}(x)|\right\}dx\\
&= \frac{1}{2(b-a)}\int_a^b\left\{|\mu^\delta_{\widetilde{A}}(x)-\mu^\delta_{\widetilde{B}}(x)|+|\nu^\delta_{\widetilde{A}}(x)-\nu^\delta_{\widetilde{B}}(x)|+\right.\\
&\qquad \left.+|(1-\mu^\delta_{\widetilde{A}}(x)-\nu^\delta_{\widetilde{A}}(x))-(1-\mu^\delta_{\widetilde{B}}(x)-\nu^\delta_{\widetilde{B}}(x))|\right\}dx\\
&= \frac{1}{2(b-a)}\int_a^b\left\{|\mu^\delta_{\widetilde{A}}(x)-\mu^\delta_{\widetilde{B}}(x)|+|\nu^\delta_{\widetilde{A}}(x)-\nu^\delta_{\widetilde{B}}(x)|+\right.\\
&\qquad \left.+|\mu^\delta_{\widetilde{A}}(x)-\mu^\delta_{\widetilde{B}}(x)+\nu^\delta_{\widetilde{A}}(x)-\nu^\delta_{\widetilde{B}}(x)|\right\}dx.
\end{aligned}$$

Since the $\delta$-fuzzy condition $0\leqslant\mu^\delta+\nu^\delta\leqslant 1$ is fulfilled, we have that the last modular expression $|\mu^\delta_{\widetilde{A}}-\mu^\delta_{\widetilde{B}}+\nu^\delta_{\widetilde{A}}-\nu^\delta_{\widetilde{B}}|=|\mu^\delta_{\widetilde{A}}+\nu^\delta_{\widetilde{A}}-(\mu^\delta_{\widetilde{B}}+\nu^\delta_{\widetilde{B}})|\leqslant 1$ for all $x$. So, the last integrand function has the form $f(x,y)=|x|+|y|+|x+y|$, where $|x|\leqslant 1$, $|y|\leqslant 1$, and $|x+y|\leqslant 1$, which reaches the maximum value $f(x,y)=2$ just on the boundary of its region of definition. Therefore, it is finally obtained that $d(\widetilde{A},\widetilde{B})\leqslant\frac{1}{2(a-b)}\int_a^b 2dx=\frac{1}{2(b-a)}2(b-a)=1$. □

Let us denote $\Pi_{\widetilde{A}}=\frac{1}{(b-a)}\int_a^b\pi^\delta_{\widetilde{A}}(x)dx$. The following theorem reveals that the distance between full security and full insecurity exceeds 1 in a measure of uncertainty, precisely in an amount equal to $\Pi_{\widetilde{A}}$.

**Theorem 6.** *For all $\delta$-fuzzy sets $\widetilde{A}\in\widetilde{\mathcal{P}}(X)$, $X=[a,b]\subset\mathbb{R}$, it is fulfilled that $d(\widetilde{A},\widetilde{\mathcal{O}})+d(\widetilde{A},\widetilde{\mathcal{I}})=1+\Pi_{\widetilde{A}}$.*

*Proof.* We have

$$\begin{aligned}
d(\widetilde{A},\widetilde{\mathcal{O}}) &= \frac{1}{2(b-a)}\int_a^b\left\{|\mu^\delta_{\widetilde{A}}(x)-0|+|\nu^\delta_{\widetilde{A}}(x)-1|+|1-\mu^\delta_{\widetilde{A}}(x)-\nu^\delta_{\widetilde{A}}(x)-0|\right\}dx\\
&= \frac{1}{2(b-a)}\int_a^b\left\{\mu^\delta_{\widetilde{A}}(x)+1-\nu^\delta_{\widetilde{A}}(x)+1-\mu^\delta_{\widetilde{A}}(x)-\nu^\delta_{\widetilde{A}}(x)\right\}dx\\
&= \frac{1}{2(b-a)}\int_a^b\left\{2-2\nu^\delta_{\widetilde{A}}(x)\right\}dx\\
&= 1-\frac{1}{(b-a)}\int_a^b\nu^\delta_{\widetilde{A}}(x)dx.
\end{aligned}$$

Similarly it turns out that $d(\widetilde{A},\widetilde{\mathcal{I}})=1-\frac{1}{(b-a)}\int_a^b\mu^\delta_{\widetilde{A}}(x)dx$. Hence, $d(\widetilde{A},\widetilde{\mathcal{O}})+d(\widetilde{A},\widetilde{\mathcal{I}})=2-\frac{1}{(b-a)}\int_a^b\{\mu^\delta_{\widetilde{A}}(x)+\nu^\delta_{\widetilde{A}}(x)\}dx=2-\frac{1}{(b-a)}\int_a^b\{1-\pi^\delta_{\widetilde{A}}(x)\}dx=1+\frac{1}{(b-a)}\int_a^b\pi^\delta_{\widetilde{A}}(x)dx=1+\Pi_{\widetilde{A}}$. □

**Theorem 7.** *Let $\widetilde{A} \prec \widetilde{B}$, two $\delta$-fuzzy sets defined on the interval $X = [a,b] \subset \mathbb{R}$. So, it turns out the following*

*a)* $d(\widetilde{A},\widetilde{\mathcal{I}}) \geqslant d(\widetilde{B},\widetilde{\mathcal{I}})$

*b)* $d(\widetilde{A},\widetilde{\mathcal{O}}) \leqslant d(\widetilde{B},\widetilde{\mathcal{O}})$

*c)* $d(\widetilde{B},\widetilde{\mathcal{I}}) - d(\widetilde{A},\widetilde{\mathcal{I}}) \leqslant \Pi_{\widetilde{B}} - \Pi_{\widetilde{A}} \leqslant d(\widetilde{B},\widetilde{\mathcal{O}}) - d(\widetilde{A},\widetilde{\mathcal{O}})$.

*Proof.* If $\widetilde{A} \prec \widetilde{B}$, then exists $\widetilde{C}$ so that $\widetilde{A} \oplus \widetilde{C} = \widetilde{B}$. Therefore, $\mu_{\widetilde{A}\oplus\widetilde{C}} = \mu_{\widetilde{B}}$ and $\nu_{\widetilde{A}\oplus\widetilde{C}} = \nu_{\widetilde{B}}$. For the case of inequality (a), we have $\mu_{\widetilde{A}\oplus\widetilde{C}} = (\mu^{\delta}_{\widetilde{A}} + \mu^{\delta}_{\widetilde{C}} - \mu^{\delta}_{\widetilde{A}}\mu^{\delta}_{\widetilde{C}})^{\frac{1}{\delta}}$, then $\mu^{\delta}_{\widetilde{A}} + \mu^{\delta}_{\widetilde{C}} - \mu^{\delta}_{\widetilde{A}}\mu^{\delta}_{\widetilde{C}} = \mu^{\delta}_{\widetilde{B}}$ which is equivalent to $(1-\mu^{\delta}_{\widetilde{A}})(1-\mu^{\delta}_{\widetilde{C}}) = 1-\mu^{\delta}_{\widetilde{B}}$. Since $\mu^{\delta}_{\widetilde{A}}, \mu^{\delta}_{\widetilde{B}}, \mu^{\delta}_{\widetilde{C}} \in [0,1]$, it follows that $\mu^{\delta}_{\widetilde{A}}(x) \leqslant \mu^{\delta}_{\widetilde{B}}(x)$, for all $x \in [a,b]$. Hence,

$$1 - \frac{1}{(b-a)}\int_a^b \mu^{\delta}_{\widetilde{A}}(x)dx \geqslant 1 - \frac{1}{(b-a)}\int_a^b \mu^{\delta}_{\widetilde{B}}(x)dx$$

which means $d(\widetilde{A},\widetilde{\mathcal{I}}) \geqslant d(\widetilde{B},\widetilde{\mathcal{I}})$. Correspondingly, inequality (b) holds.

For the case of (c), we first apply Theorem 6 for both sets $\widetilde{A}$ and $\widetilde{B}$. From this it follows

$$\Pi_{\widetilde{B}} - \Pi_{\widetilde{A}} = [d(\widetilde{B},\widetilde{\mathcal{O}}) - d(\widetilde{A},\widetilde{\mathcal{O}})] - [d(\widetilde{A},\widetilde{\mathcal{I}}) - d(\widetilde{B},\widetilde{\mathcal{I}})].$$

From (a) and (b) it follows that both expressions in square brackets are non-negative. From here both inequalities of (c) follow, which concludes the proof. □

# 4 A new approach to the $\delta$-fuzzy mean concept

Defining a mean is not always an easy task. An illustrative example consists of the problem of determining the geometric mean between two matrices (Lawson & Lim, 2001). To introduce the concept of $\delta$-fuzzy mean, we will follow an axiomatic path. For this we will require the following property.

**Theorem 8.** *Let $\widetilde{A}_1 = \langle \mu_1, \nu_1\rangle, \widetilde{A}_2 = \langle \mu_2, \nu_2\rangle, ..., \widetilde{A}_n = \langle \mu_n, \nu_n\rangle \in \widetilde{\mathcal{P}}(X)$. We have*

$$\bigoplus_{i=1}^{n} \widetilde{A}_i = \left\langle \left[1 - \prod_{i=1}^{n}(1-\mu_i^{\delta})\right]^{\frac{1}{\delta}}, \prod_{i=1}^{n}\nu_i \right\rangle.$$

*Proof.* We will use the principle of complete induction. For $n = 1$ (basis step) we have $\langle [1-(1-\mu_1^{\delta})]^{\frac{1}{\delta}}, \nu_1\rangle = \langle \mu_1, \nu_1\rangle = \widetilde{A}_1$. Assume that the statement holds for some integer $n = k \geqslant 1$ (inductive step). Thus we have

$$\begin{aligned}
\bigoplus_{i=1}^{k+1} \widetilde{A}_i &= \Big(\bigoplus_{i=1}^{k} \widetilde{A}_i\Big) \oplus \widetilde{A}_{k+1} \\
&= \left\langle \left[1 - \prod_{i=1}^{k}(1-\mu_i^{\delta})\right]^{\frac{1}{\delta}}, \prod_{i=1}^{k}\nu_i \right\rangle \oplus \langle \mu_{k+1}, \nu_{k+1}\rangle \\
&= \left\langle \left\{1 - \prod_{i=1}^{k}(1-\mu_i^{\delta}) + \mu_{k+1}^{\delta} - \left[1 - \prod_{i=1}^{k}(1-\mu_i^{\delta})\right]\mu_{k+1}^{\delta}\right\}^{\frac{1}{\delta}}, \Big(\prod_{i=1}^{k}\nu_i\Big)\nu_{k+1} \right\rangle \\
&= \left\langle \left[1 - (1-\mu_{k+1}^{\delta})\prod_{i=1}^{k}(1-\mu_i^{\delta})\right]^{\frac{1}{\delta}}, \prod_{i=1}^{k+1}\nu_i \right\rangle \\
&= \left\langle \left[1 - \prod_{i=1}^{k+1}(1-\mu_i^{\delta})\right]^{\frac{1}{\delta}}, \prod_{i=1}^{k+1}\nu_i \right\rangle.
\end{aligned}$$

Therefore, the statement is also true for $n = k + 1$. This completes the proof. $\square$

In particular, the case $\widetilde{A}_1 = \widetilde{A}_2 = ... = \widetilde{A}_n = \widetilde{A}$ produces the expression corresponding to Definition 4, for $r \in \mathbb{Z}_{\geqslant 1}$. This fact also suggests that such a definition is reasonably consistent. The axiomatic approach to averages dates back to the memoirs published by J. V. Schiaparelli in the second half of the 19th century (Schiaparelli, 1875, Observatoire de Bréra à Milan). According to Bonferroni, a mean $M$ of the quantities $x_1, x_2, ..., x_n$, with respective weights $P_1, P_2, ..., P_n$, is expressed by the quasi-arithmetic relation (Bonferroni, 1924):

$$\psi(M) = \frac{P_1\psi(x_1) + ... + P_n\psi(x_n)}{P_1 + ... + P_n}.$$

where the quantities $x_i$ are different from each other, the weights $P_i$ are all positive, and the function $\psi$ is continuous and strictly increasing. The convenient selection of $\psi$ produces commonly used averages. For instance, $\psi = x$ gives the weighted arithmetic mean, $\psi = \log x$ $(x > 0)$ the weighted geometric mean, $\psi = \frac{1}{x}$ $(x > 0)$ the weighted harmonic mean, and so on. There are several ways to characterize the concept of the mean $M = M(x_1, ..., x_n)$ (Muliere & Parmigiani, 1993). For example, the following four requirements are relatively intuitive:

*a*) continuity and strict monotony in each variable $x_i$'s,

*b*) symmetry, i.e, invariance to labeling of the $x_i$'s,

*c*) reflexivity, i.e, when all the $x_i$'s are equal to the same value, that value is the mean $M(x, ..., x) = x$,

*d*) associativity, in the sense of a subset of values can be replaced by their mean with no effect on the total mean $M(x_1, ..., x_n) = M(x, ..., x, x_{k+1}, ..., x_n)$, where $x = M(x_1, ..., x_k)$ and $1 < k < n - 1, n \geqslant 2$.

Independently, Kolmogorov and Nagumo showed that if the conditions (a)-(d) holds, then the mean $M(x_1, ..., x_n)$ has adopt the Manferroni's shape for weighting factors $P_i = 1$ (Kolmogorov, 1930; Nagumo, 1930).

**Theorem 9** (Kolmogorov-Nagumo)**.** *Let $I \in \mathbb{R}$ be a closed and bounded interval and let the application $M : \bigcup_{n=1}^{\infty} I^n \to \mathbb{R}$. Conditions (a)-(d) hold if and only if there exists a function $\psi$, strictly monotonic and continuous, such that*

$$M(x_1, ..., x_n) = \psi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\psi(x_i)\right).$$

Since $(\widetilde{\mathcal{P}}(X), \oplus, *)$ is a right $\mathbb{R}_{\geqslant 0}$-semimodule, it is possible to consider multiplication by $\frac{1}{n} \in \mathbb{R}_{>0}$ as a particular case of dot product, the $\delta$-fuzzy sum of each set $\psi(\widetilde{A}_i)$, as a function of the form $\psi(\langle \mu_i, \nu_i \rangle) = \langle \psi_\mu(\mu_i), \psi_\nu(\nu_i) \rangle$, where $\psi_\mu : [0,1] \to [0,1]$ and $\psi_\nu : [0,1] \to [0,1]$ are continuous and strictly monotonous applications. In this way, we can transfer the notion of $\delta$-fuzzy mean $\widetilde{M}$ to each component of a $\delta$-fuzzy set. Effectively, if in addition each $\psi(\widetilde{A}_i)$ is a $\delta$-fuzzy for all $i = 1, 2, ..., n$, then it follows that

$$\begin{aligned} \widetilde{M} &= \psi^{-1}\Big(\frac{1}{n}\bigoplus_{i=1}^{n}\psi(\widetilde{A}_i)\Big) \\ &= \psi^{-1}\Big(\frac{1}{n}\bigoplus_{i=1}^{n}\Big\langle \psi_\mu(\mu_i), \psi_\nu(\nu_i)\Big\rangle\Big) \end{aligned}$$

$$
\begin{aligned}
&= \psi^{-1}\Big(\frac{1}{n}\Big\langle\Big\{1-\prod_{i=1}^{n}[1-\psi_\mu^\delta(\mu_i)]\Big\}^{\frac{1}{\delta}}, \prod_{i=1}^{n}\psi_\nu(\nu_i)\Big\rangle\Big) \text{ (by Theorem 8).}\\
&= \psi^{-1}\Big(\Big\langle\Big\{1-\Big[1-\Big(1-\prod_{i=1}^{n}[1-\psi_\mu^\delta(\mu_i)]\Big)\Big]^{\frac{1}{n}}\Big\}^{\frac{1}{\delta}}, \prod_{i=1}^{n}\psi_\nu^{\frac{1}{n}}(\nu_i)\Big\rangle\Big) \text{ (by Definition 4)}\\
&= \psi^{-1}\Big(\Big\langle\Big\{1-\prod_{i=1}^{n}[1-\psi_\mu^\delta(\mu_i)]^{\frac{1}{n}}\Big\}^{\frac{1}{\delta}}, \prod_{i=1}^{n}\psi_\nu^{\frac{1}{n}}(\nu_i)\Big\rangle\Big)\\
&= \Big\langle\psi_\mu^{-1}\Big(\Big\{1-\prod_{i=1}^{n}[1-\psi_\mu^\delta(\mu_i)]^{\frac{1}{n}}\Big\}^{\frac{1}{\delta}}\Big), \psi_\nu^{-1}\Big(\prod_{i=1}^{n}\psi_\nu^{\frac{1}{n}}(\nu_i)\Big)\Big\rangle.
\end{aligned}
$$

For example, a kind of natural average is obtained when $\psi(x,y)=(x,y)=\psi^{-1}(x,y)$, that is,

$$\widetilde{M}_{nat} = \Big\langle\Big\{1-\sqrt[n]{\textstyle\prod_{i=1}^{n}(1-\mu_i^\delta)}\Big\}^{\frac{1}{\delta}}, \sqrt[n]{\textstyle\prod_{i=1}^{n}\nu_i}\Big\rangle.$$

On the other hand, if we take

$$\psi(x,y)=\Big([1-\exp(-x^p)]^{\frac{1}{\delta}}, \exp\big(-\frac{1}{k}y^p\big)\Big),$$

with $k=\delta \min\limits_{1\leqslant i\leqslant n}\nu_i^p$ and $p>0$, then

$$
\begin{aligned}
\big\{[1-\exp(-\mu_i^p)]^{\frac{1}{\delta}}\big\}^\delta + \Big[\exp\big(-\frac{1}{k}\nu_i^p\big)\Big]^\delta &= 1-\exp(-\mu_i^p)+\exp\Big(-\frac{\nu_i^p}{\min\limits_{1\leqslant i\leqslant n}\nu_i^p}\Big)\\
&\leqslant 1-\exp(-1)+\exp(-1)\\
&= 1.
\end{aligned}
$$

for all $i=1,2,...,n$ and consequently each $\psi(\widetilde{A}_i)$ is a $\delta$-fuzzy set. Likewise, it turns out the same outcome if we adopt $k=\delta\max\limits_{1\leqslant i\leqslant n}\nu_i^p$ for $p<0$. Since $\psi^{-1}(x,y)=\big([-\ln(1-x^\delta)]^{\frac{1}{p}}, (-k\ln y)^{\frac{1}{p}}\big)$, then it follows the following power mean for $p\in\mathbb{R}^*$:

$$\widetilde{M}_p = \Big\langle\Big(\frac{1}{n}\sum_{i=1}^{n}\mu_i^p\Big)^{\frac{1}{p}}, \Big(\frac{1}{n}\sum_{i=1}^{n}\nu_i^p\Big)^{\frac{1}{p}}\Big\rangle.$$

Under this approach, $\widetilde{M}_1$ is structured by arithmetic means, $\widetilde{M}_2$ by quadratic means, $\widetilde{M}_{-1}$ by harmonic means, and so on. Particularly, $\widetilde{M}_2$ matches the weighted power mean aggregation operator in Pythagorean fuzzy sets, taking each weight $w_i=\frac{1}{n}$, for $1\leqslant i\leqslant n$ (Yager & Abbasov, 2013). It is also well known that the geometric mean is obtained in the limiting case when $p\to 0$ (see, e.g., Hardy, Littlewood, & Pólya, 1934). By this way it is obtained that $\widetilde{M}_p\to\widetilde{M}_0=\big\langle\sqrt[n]{\prod_{i=1}^{n}\mu_i}, \sqrt[n]{\prod_{i=1}^{n}\nu_i}\big\rangle$, which coincides particularly with the operator $\$_{i=1}^{n}\widetilde{A}_i$ (see Definition 7(ii), Jamkhaneh & Garg, 2018), structured by two geometric means.

In dual mode, it is usual to define the union $\cup_{i=1}^{n}\widetilde{A}_i=\langle\max_{1\leqslant i\leqslant n}\mu_i, \min_{1\leqslant i\leqslant n}\nu_i\rangle$ and the intersection $\cap_{i=1}^{n}\widetilde{A}_i=\langle\min_{1\leqslant i\leqslant n}\mu_i, \max_{1\leqslant i\leqslant n}\nu_i\rangle$, in intuitionistic fuzzy sets (Atanassov & Vassilev, 2018), in Pythagoreans (Peng & Yang, 2015), in those of the third type (Begum & Srinivasan, 2017), and even in the general $\delta$-fuzzy case (Jamkhaneh & Garg, 2018). Now we have $\widetilde{M}_{+\infty}=\lim_{p\to+\infty}\widetilde{M}_p=\big\langle\max_{1\leqslant i\leqslant n}\mu_i, \max_{1\leqslant i\leqslant n}\nu_i\big\rangle$ and $\widetilde{M}_{-\infty}=\lim_{p\to-\infty}\widetilde{M}_p=\big\langle\min_{1\leqslant i\leqslant n}\mu_i, \min_{1\leqslant i\leqslant n}\nu_i\big\rangle$. Last observation about the duals $\widetilde{M}_{+\infty}$ and $\widetilde{M}_{-\infty}$ provides a complementary notion, where pairs of maxima and pairs of minima also make sense.

After the analysis performed, it remains to verify that $\widetilde{M}$ is a $\delta$-fuzzy set. For example, it is obvious that $\widetilde{M}_{+\infty}$ is not necessarily $\delta$-fuzzy in a general way. The case $\widetilde{M}_{nat}$ is verified directly from the conditions $\nu_i^\delta \leqslant 1-\mu_i^\delta$, for $i=1,2,...,n$. Let us see below that $\widetilde{M_p}$ is a $\delta$-fuzzy set for $0<p<\delta$. Indeed, this results in $\frac{\delta}{p}>1$ and thus we can apply the Second Minkowski's inequality (Cvetkovski, 2012, Theorem 9.6, p. 99) as follows:

$$\begin{aligned}
\left[\Big(\frac{1}{n}\sum_{i=1}^{n}\mu_i^p\Big)^{\frac{1}{p}}\right]^{\delta}+\left[\Big(\frac{1}{n}\sum_{i=1}^{n}\nu_i^p\Big)^{\frac{1}{p}}\right]^{\delta} &= \Big(\frac{1}{n}\Big)^{\frac{\delta}{p}}\left[\Big(\sum_{i=1}^{n}\mu_i^p\Big)^{\frac{\delta}{p}}+\Big(\sum_{i=1}^{n}\nu_i^p\Big)^{\frac{\delta}{p}}\right] \\
&\leqslant n^{-\frac{\delta}{p}}\left\{\sum_{i=1}^{n}\left[\big(\mu_i^p\big)^{\frac{\delta}{p}}+\big(\nu_i^p\big)^{\frac{\delta}{p}}\right]^{\frac{p}{\delta}}\right\}^{\frac{\delta}{p}} \\
&= n^{-\frac{\delta}{p}}\left[\sum_{i=1}^{n}\Big(\mu_i^\delta+\nu_i^\delta\Big)^{\frac{p}{\delta}}\right]^{\frac{\delta}{p}} \\
&\leqslant n^{-\frac{\delta}{p}}\Big(\sum_{i=1}^{n}1\Big)^{\frac{\delta}{p}} \\
&= 1.
\end{aligned}$$

Based on the above considerations, it is possible to define the concept of $\delta$-fuzzy mean as follows:

**Definition 8.** *Let $\widetilde{A}_1=\langle\mu_1,\nu_1\rangle,\widetilde{A}_2=\langle\mu_2,\nu_2\rangle,...,\widetilde{A}_n=\langle\mu_n,\nu_n\rangle\in\widetilde{\mathcal{P}}(X)$. Let $\psi:\widetilde{\mathcal{P}}(X)\to\widetilde{\mathcal{P}}(X)$ be defined by $\psi(\langle x,y\rangle)=\langle\psi_\mu(x),\psi_\nu(y)\rangle$, such that $\psi_\mu:[0,1]\to[0,1]$ and $\psi_\nu:[0,1]\to[0,1]$ are continuous and strictly monotonous functions, so that $\psi(\widetilde{A}_i)\in\widetilde{\mathcal{P}}(X)$ for all $i=1,2,...,n$. Then, $\widetilde{M}$ is a $\delta$-fuzzy mean of $\widetilde{A}_1,\widetilde{A}_2,...,\widetilde{A}_n$ iff it has the form*

$$\left\langle \psi_\mu^{-1}\left(\Big\{1-\prod_{i=1}^{n}[1-\psi_\mu^\delta(\mu_i)]^{\frac{1}{n}}\Big\}^{\frac{1}{\delta}}\right),\psi_\nu^{-1}\left(\prod_{i=1}^{n}\psi_\nu^{\frac{1}{n}}(\nu_i)\right)\right\rangle$$

*and also it is a $\delta$-fuzzy set.*

As pointed out by Marichal (Marichal, 2000), for real increasing means, the Kolmogorov-Nagumo reflexive property is equivalent to the Cauchy internality (Cauchy, 1821). Internality requires a total order because of this, in A-IFS model, a total order has been defined relating the aggregation functions with score and accuracy functions (Xu & Yager, 2006). This approach has raised obstacles related to monotony, which has recently been overcome by employing the Łukasiewicz triangular norm (Wang & Mendel, 2019). In our approach, the binary relation $\prec$ is a preorder, compatible with $(\widetilde{\mathcal{P}}(X),\oplus,*)$ as right $\mathbb{R}_{\geqslant 0}$-semimodule. We do not have a total order, however the following boundedness property for a $\delta$-fuzzy mean subsists.

**Theorem 10.** *Let $\widetilde{M}$ a $\delta$-fuzzy mean of $\widetilde{A}_1,\widetilde{A}_2,...,\widetilde{A}_n\in\widetilde{\mathcal{P}}(X)$, where $X=[a,b]\subset\mathbb{R}$. It is fulfilled that*

$$\min_{1\leqslant i\leqslant n}\big\{d\big(\widetilde{A}_i,\widetilde{\mathcal{O}}\big)\big\}\leqslant d\big(\widetilde{M},\widetilde{\mathcal{O}}\big)\leqslant\max_{1\leqslant i\leqslant n}\big\{d\big(\widetilde{A}_i,\widetilde{\mathcal{O}}\big)\big\}.$$

*Proof.* We know that $d(\widetilde{A},\widetilde{\mathcal{O}})=1-\frac{1}{(b-a)}\int_a^b\nu_{\widetilde{A}}^\delta(x)dx$. Applying this identity for $\widetilde{A}=\widetilde{A}_i$ $(1\leqslant i\leqslant n)$ and $\widetilde{A}=\widetilde{M}$, we get

$$1-\frac{1}{(b-a)}\int_a^b\max_{1\leqslant i\leqslant n}\nu_i^\delta(x)dx\leqslant 1-\frac{1}{(b-a)}\int_a^b\nu_{\widetilde{M}}^\delta(x)dx\leqslant 1-\frac{1}{(b-a)}\int_a^b\min_{1\leqslant i\leqslant n}\nu_i^\delta(x)dx$$

i.e.,

$$\int_a^b\min_{1\leqslant i\leqslant n}\nu_i^\delta(x)dx\leqslant\int_a^b\nu_{\widetilde{M}}^\delta(x)dx\leqslant\int_a^b\max_{1\leqslant i\leqslant n}\nu_i^\delta(x)dx$$

which is obvious since $\delta>0$ and $\min_{1\leqslant i\leqslant n}\nu_i(x)\leqslant\nu_{\widetilde{M}}(x)\leqslant\max_{1\leqslant i\leqslant n}\nu_i(x)$ for all $x\in[a,b]$. $\square$

We can see that in the monoid $(\mathbb{R}_{\geqslant 0}, +)$ it is permissible to define $a \leqslant b$ iff $d(a, 0) \leqslant d(b, 0)$, where $d(a, b) = |a - b|$. Now, analogously to the previous result, for a sequence of real and non-negative numbers $a_1, a_2, ..., a_n$, it is true that $\min_{1 \leqslant i \leqslant n}\{|a_i - 0|\} \leqslant |M(a_1, a_2, ..., a_n) - 0| \leqslant \max_{1 \leqslant i \leqslant n}\{|a_i - 0|\}$. Which is another way of expressing the internality of a real mean $M(a_1, a_2, ..., a_n)$, but in the language of distance.

# 5 An application example

Next, we present an application example. A panel of 10 experts responds by means of two scales made up of integers from 0 to 10, to what extent they agree or disagree, with regard to the year corresponding to the period 2026-2030, where it is convenient to make a certain business investment. In this strategic forecasting, the value 10 indicates that the expert completely agrees, while 0 means totally disagree. The five-year period analyzed is reasonable and it is expected that the experts will not select extreme values. All the data are normalized, dividing the reactions of each expert by 10. As usually happens in this type of study, the experts do not always respond $1 - p$ to the statement $\neg A$, when they have previously responded $p$ to the statement $A$ (Cruz & Cables, 2022). The Table 1 contains the results of the empirical study carried out. Obviously, most of the data do not fit the intuitionistic fuzzy model, since the

**Table 1:** Three $\delta$-fuzzy means in expert opinion processing

| | 2026 | | 2027 | | 2028 | | 2029 | | 2030 | |
|---|---|---|---|---|---|---|---|---|---|---|
| Expert | $\mu_i$ | $\nu_i$ | $\mu_i$ | $\nu_i$ | $\mu_i$ | $\nu_i$ | $\mu_i$ | $\nu_i$ | $\mu_i$ | $\nu_i$ |
| 1 | 0.4 | 0.6 | 0.1 | 0.8 | 0.9 | 0.4 | 0.5 | 0.5 | 0.7 | 0.6 |
| 2 | 0.9 | 0.3 | 0.3 | 0.8 | 0.9 | 0.3 | 0.2 | 0.2 | 0.7 | 0.7 |
| 3 | 0.4 | 0.9 | 0.7 | 0.6 | 0.9 | 0.1 | 0.5 | 0.5 | 0.4 | 0.8 |
| 4 | 0.6 | 0.4 | 0.7 | 0.5 | 0.4 | 0.2 | 0.5 | 0.7 | 0.5 | 0.8 |
| 5 | 0.1 | 0.1 | 0.2 | 0.1 | 0.5 | 0.3 | 0.6 | 0.8 | 0.1 | 0.4 |
| 6 | 0.8 | 0.5 | 0.3 | 0.5 | 0.4 | 0.4 | 0.6 | 0.7 | 0.8 | 0.3 |
| 7 | 0.5 | 0.7 | 0.6 | 0.2 | 0.8 | 0.2 | 0.7 | 0.6 | 0.5 | 0.8 |
| 8 | 0.5 | 0.7 | 0.3 | 0.6 | 0.9 | 0.6 | 0.5 | 0.3 | 0.2 | 0.7 |
| 9 | 0.3 | 0.7 | 0.6 | 0.6 | 0.7 | 0.2 | 0.4 | 0.6 | 0.4 | 0.7 |
| 10 | 0.2 | 0.5 | 0.3 | 0.5 | 0.5 | 0.4 | 0.6 | 0.3 | 0.3 | 0.5 |
| $\widetilde{M}_{nat}$ | 0.62 | 0.47 | 0.51 | 0.45 | 0.79 | 0.28 | 0.54 | 0.48 | 0.57 | 0.60 |
| $\widetilde{M}_1$ | 0.47 | 0.54 | 0.41 | 0.52 | 0.69 | 0.31 | 0.51 | 0.52 | 0.46 | 0.63 |
| $\widetilde{M}_2$ | 0.53 | 0.58 | 0.46 | 0.56 | 0.72 | 0.34 | 0.53 | 0.55 | 0.51 | 0.65 |

respective sums in the responses are generally different than 1. The Pythagorean fuzzy model fits the most, with the exception of expert Nº 8 for the year 2028, where $0.9^2 + 0.6^2 > 1$. The minimum value of $\delta$, such that $0 \leqslant \delta \leqslant 1$, is $\delta = 3$. Therefore, it is necessary to average using means suitable for a 3-fuzzy model, such as $\widetilde{M}_{nat}$ and $\widetilde{M}_p$, where $0 < p < 3$.

The last three rows in Table 1 show the results of the calculations for $\widetilde{M}_{nat}$, $\widetilde{M}_1$, and $\widetilde{M}_2$. The most convenient year is 2028, where the panel of experts has a higher level of agreement and a lower level of disagreement, in all the calculated means. In the case of $\widetilde{M}_1$, the uncertainty levels are the highest (closer to 1). For $\widetilde{M}_{nat}$ and $\widetilde{M}_2$, the year of least uncertainty is also 2028 ($\pi = 0.79$ and $\pi = 0.84$, respectively).

# 6 Conclusion

The present investigation deals with a $\delta$-fuzzy generalization of the A-IFS model, based on the underlying algebraic structure. We show that $(\widetilde{\mathcal{P}}(X), \oplus, *, \prec)$ construct is a pre-ordered and bounded left $\mathbb{R}_{\geqslant 0}$-semimodule. This algebraic approach has allowed us to discuss certain metric properties and also to show a new way to define the notion of fuzzy mean. For a new study it is suggestive to explore the quotient set that results from the relationship between sets with equal distance from $\widetilde{\mathcal{O}}$, which produces a poset. On the other hand, it is also possible to apply these results in studies related to multi-criteria decision-making problems, where it is convenient to extend the analysis to weighted $\delta$-fuzzy means. From a practical point of view, this research also illustrates how to select an appropriate $\delta$-fuzzy mean, depending on the fuzzy model that best fits experimental data.

# 7 Acknowledgement

The work is supported by the University of Holguín (Cuba) and the National University of the Northeast (Argentina).

# References

Atanassov, K.T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *20*(1), 87-96. doi: 10.1016/S0165-0114(86)80034-3

Atanassov, K.T. (2012). *On Intuitionistic Fuzzy Sets Theory*. Berlin: Springer-Verlag. doi: 10.1007/978-3-642-29127-2

Atanassov, K.T., Vassilev, P. (2018). On the intuitionistic fuzzy sets of $n$-th type. In A.E. Gawęda et al. (eds.), *Advances in Data Analysis with Computational Intelligence Methods*, Studies in Computational Intelligence, *738*, Cham: Springer. doi: 10.1007/978-3-319-67946-4_10

Begum, S.S., Srinivasan, R. (2017). Some properties of intuitionistic fuzzy sets of third type. *Advances in Fuzzy Mathematics*, *12*(2), 189-195. https://www.ripublication.com/afm17/afmv12n2_03.pdf

Bonferroni, C.E. (1924). La media esponenziale in matematica finanziaria. *Annuario del Regio Istituto Superiore di Scienze Economiche e Commerciali di Bari*, *1923-1924*, 1-14. https://www.ripublication.com/afm17/afmv12n2_03.pdf

Bustince, H., Barrenechea, E., Pagola, M., Fernandez, J., Xu, Z., Bedregal, B., Montero, J., Hagras, H., Herrera, F., & De Baets, B. (2016). A historical account of types of fuzzy sets and their relationships. *IEEE Transactions on Fuzzy Systems*, *24*(1), 179-194. doi: 10.1109/TFUZZ.2015.2451692

Cauchy, A.-L. (1821). *Cours d'Analyse de l'École Royale Polytechnique. I.$^{re}$ Partie. Analyse Algébrique*. Paris: Debure.

Cruz, M., Cables, E.H. (2021). Una generalización del Delphi difuso para estudios prospectivos. *Universidad y Sociedad*, *13*(2), 57-66. https://rus.ucf.edu.cu/index.php/rus/article/view/1942/1933

Cruz, M., Cables, E.H. (2022). Un estudio comparado de dos enfoques no deterministas en el Delphi de pronóstico. *Investigación Operacional*, *43*(2), 102-119. http://rev-inv-ope.pantheonsorbonne.fr/sites/default/files/inline-files/43122-09.pdf

Cvetkovski, Z. (2012). *Inequalities. Theorems, Techniques and Selected Problems*. Berlin: Springer-Verlag. doi: 10.1007/978-3-642-23792-8

Deschrijver, G., Kerre, E.F. (2002). A generalization of operators on intuitionistic fuzzy sets using triangular norms and conorms. *Notes on Intuitionistic Fuzzy Sets*, *8*(1), 19–27. http://ifigenia.org/images/b/b5/NIFS-08-1-19-27.pdf

Ejegwa, P.A. (2018). Distance and similarity measures for Pythagorean fuzzy sets. *Granular Computing*, *5*, 225-238. doi: 10.1007/s41066-018-00149-z

Golan, J.S. (2003). *Semirings and Affine Equations over Them: Theory and Applications*. Dordrecht: Springer. doi: 10.1007/978-94-017-0383-3

Gondran, M., Minoux, M. (1984). Linear algebra in dioids: a survey of recent results. *Annals of Discrete Mathematics*, *19*, 147-164. doi: 10.1016/S0304-0208(08)72960-8

Hardy, G.H., Littlewood, J.E., & Pólya, G. (2012). *Inequalities*. Cambridge: Cambridge University Press.

Jamkhaneh, E.B., Garg, H. (2018). Some new operations over the generalized intuitionistic fuzzy sets and their application to decision-making process. *Granular Computing*, *3*(2), 111-122. doi: 10.1007/s41066-017-0059-0

Jamkhaneh, E.B., Nadarajah, S. (2015). A new generalized intuitionistic fuzzy set. *Hacettepe Journal of Mathematics and Statistics*, *44*(6), 1537-1551. doi: 10.15672/HJMS.2014367557

Kolmogorov, A.N. (1930). Sur la notion de la moyenne. *Atti Accad. Naz. Lincei Rend*, *12*(9), 388-391. English translation in V. M. Tikhomirov (1991, Ed.), *Selected Works of A. N. Kolmogorov*, Vol. I. Mathematics and Mechanic (pp. 144-146). Kluwer Academic Publishers.

Lawson, J.D., Lim, Y. (2004). The geometric mean, matrices, metrics, and more. *The American Mathematical Monthly*, *108*(9), 797-812. doi: 10.1080/00029890.2001.11919815

Marichal, J.-L. (2004). On an axiomatization of the quasi-arithmetic mean values without the symmetry axiom. *Aequationes Mathematicae*, *59*(1), 74-83. doi: 10.1007/PL00000129

Milošević, P., Petrović, B., & Dragović, I. (2021). A novel approach to generalized intuitionistic fuzzy sets based on interpolative boolean algebra. *Mathematics*, *9*(17), 2115. doi: 10.3390/math9172115

Muliere, P., Parmigiani, G. (1993). Utility and means in the 1930s. *Statistical Science*, *8*(4), 421-432. doi: 10.1214/ss/1177010786

Nagumo, M. (1930). Über eine Klasse der Mittelwerte. *Japanese Journal of Mathematics: Transactions and Abstracts*, *7*, 71-79. doi: 10.4099/jjm1924.7.0_71

Peng, X., Yang, Y. (2015). Some results for pythagorean fuzzy sets. *International Journal of Intelligent Systems*, *30*(11), 1133-1160. doi: 10.1002/int.21738

Schiaparelli, J.V. (1876). Sur le principe de la moyenne arithmétique. *Astronomische Nachrichten*, *87*(2068), 55-58. doi: 10.1002/asna.18760870403

Seik, M.R., Nayak, P.K., & Pal, M. (2013). Notes on triangular intuitionistic fuzzy numbers. *International Journal of Mathematics in Operational Research*, *5*(4), 446-465. doi: 10.1504/IJMOR.2013.054730

Shabani, A., Jamkhaneh, E.B. (2014). A new generalized intuitionistic fuzzy number. *Journal of Fuzzy Set Valued Analysis*, *2014*, 1-10. doi: 10.5899/2014/jfsva-00199

Srinivasan, R., Palaniappan, N. (2012). Some operators on intuitionistic fuzzy sets of root type. *Annals of Fuzzy Mathematics and Informatics*, *4*, 377-383. http://www.afmi.or.kr/papers/2012/Vol-04_No-02/PDF/AFMI-4-2(377-383)-J-110402R1.pdf

Szmidt, E. (2014). *Distances and Similarities in Intuitionistic Fuzzy Sets.* Switzerland: Springer. doi: 10.1007/978-3-319-01640-5

Tcvetkov, T., Szmidt, E., & Kacprzyk, J. (2009). On some issues related to the distances between the Atanassov intuitionistic fuzzy sets. *Cybernetics and Information Technologies*, *9*(2), 54-61. http://www.cit.iit.bas.bg/CIT_09/v9-2/54-61.pdf

Wang, W., & Mendel, J.M. (2019). Intuitionistic fuzzy hybrid weighted arithmetic mean and its application in decision making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *27*(3), 353-373. doi: 10.1142/S0218488519500168

Yager, R.R. (2013). Pythagorean fuzzy subsets. In: *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS)* (pp. 57-61). Edmonton: IEEE. doi: 10.1109/IFSA-NAFIPS.2013.6608375

Yager, R.R. (2017). Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, *25*(5), 1222-1230. doi: 10.1109/TFUZZ.2016.2604005

Yager, R.R., & Abbasov, A.M. (2013). Pythagorean membership grades, complex numbers, and decision making. *International Journal of Intelligent Systems*, *28*(5), 436–452. doi: 10.1002/int.21584

Zadeh, L.A. (1965). Fuzzy sets. *Information and Control*, *8*(3), 338-353. doi: 10.1016/S0019-9958(65)90241-X

Zhang, X.L., Xu, Z.S. (2014). Extension of TOPSIS to multiple criteria decision making with pythagorean fuzzy sets. *International Journal of Intelligent Systems*, *29*(12), 1061-1078. doi: 10.1002/int.21676

Zu, Z., Yager, R.R. (2006). Some geometric aggregation operators based on intuitionistic fuzzy sets. *International Journal of General Systems*, *35*(4), 417-433. doi: 10.1080/03081070600574353